On automorphism groups of power semigroups over numerical semigroups

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Contents

Basic Concepts

For a semigroup S with a binary operation "$+$" (written additively):

- $\mathcal{P}_{\text{fin}}(S)$ denotes the set of all finite non-empty subsets of S.
- Binary operation:

$$X + Y = \{x + y : x \in X,\ y \in Y\}, \quad X, Y \in \mathcal{P}_{\text{fin}}(S).$$

  We call $\mathcal{P}_{\text{fin}}(S)$ the finitary power semigroup of S.
- If $0 \in S$, by $\mathcal{P}_{\text{fin},0}(S)$ we denote the monoid consisting of all finite non-empty subsets of S containing 0. This is the reduced finitary power monoid of S, with singleton $\{0\}$ as zero-element.

- $\mathbb{Z}$: additive group of all integers.
- $\mathbb{N}$: the sub-monoid of $\mathbb{Z}$ of all nonnegative integers.

Inner Automorphism

Let H be a monoid and $f \in \mathrm{Aut}(H)$. There is a natural way to extend f to an automorphism of $\mathcal{P}_{\mathrm{fin}}(H)$ (or $\mathcal{P}_{\mathrm{fin},0}(H)$):

$$X \longmapsto f[X] := \{f(x) : x \in X\}.$$

Such automorphisms, arising naturally from automorphisms of H, are called the **inner automorphisms** of $\mathcal{P}_{\mathrm{fin}}(H)$ (or $\mathcal{P}_{\mathrm{fin},0}(H)$).

Question 1

Does $\mathcal{P}_{\mathrm{fin}}(H)$ (or $\mathcal{P}_{\mathrm{fin},0}(H)$) admit any automorphism *other than* the inner ones?

A Theorem of Tringali and Yan

Theorem 1 (Tringali and Yan, 2025)

The only automorphisms of $\mathcal{P}_{fin,0}(\mathbb{N})$ *are:*

- *the* **identity** *automorphism, and*
- *the* **involution** σ_0 *defined by*

$$\sigma_0(X) \;=\; \max X \;-\; X,$$

where $\max X$ *denotes the maximum element of* X*.*

Since $\mathrm{Aut}(\mathbb{N}) = \{\mathrm{id}\}$, the involution σ_0 **cannot** be an inner automorphism of $\mathcal{P}_{\mathrm{fin}}(\mathbb{N})$.

A Conjecture

Following up on Theorem 1, Tringali and Wen [1] proved that

$$\mathrm{Aut}\big(\mathcal{P}_{\mathrm{fin}}(\mathbb{Z})\big) \;\cong\; \mathbb{Z}_2 \times \mathrm{Dih}_\infty,$$

where Dih_∞ denotes the infinite dihedral group.
This leads to a natural question:

Question 2

What is $\mathrm{Aut}\big(\mathcal{P}_{\mathrm{fin},0}(\mathbb{Z})\big)$?

Tringali and Wen proposed the following conjecture:
[Tringali and Wen] The only nontrivial automorphism of the reduced finitary power monoid of $(\mathbb{Z}, +)$ is given by

$$X \;\longmapsto\; -X \;:=\; \{-x : x \in X\}.$$

Key Observations

For any $f \in \mathrm{Aut}\big(\mathcal{P}_{\mathrm{fin},0}(\mathbb{Z})\big)$, we first examine:

$$\{0,1\}^f \longrightarrow ? \qquad \text{and} \qquad \{0,-1\}^f \longrightarrow ?$$

This is because for any $a, b \in \mathbb{N}$, we have

$$[\![-a, b]\!] = a\{0,-1\} + b\{0,1\},$$

where $[\![-a, b]\!]$ denotes the integer interval from $-a$ to b. We know that for any set $X \in \mathcal{P}_{\mathrm{fin},0}(\mathbb{Z})$, X admits the following decomposition:

$$X = [\![-a_1, b_1]\!] \cup [\![-a_2, b_2]\!] \cup [\![-a_3, b_3]\!] \cup \cdots \cup [\![-a_k, b_k]\!],$$

where k is the smallest integer for which such a decomposition exists. In this case, k is called the **boxing dimension** of X. From this decomposition, we see that $\{0,1\}$ and $\{0,-1\}$ play a crucial role — analogous to the role of a **basis** in a linear space.

A Key Lemma

First, for any $f \in \mathrm{Aut}(\mathcal{P}_{\text{fin},0}(\mathbb{Z}))$, we know $\{0\}^f = \{0\}$ since $\{0\}$ is the identity element of $\mathcal{P}_{\text{fin},0}(\mathbb{Z})$. Thus we may assume:

$$\min(\{0,1\}^f) = -a, \quad \max(\{0,1\}^f) = b, \quad \text{where } a, b \in \mathbb{N},\ a+b>0;$$

$$\min(\{0,-1\}^f) = -c, \quad \max(\{0,-1\}^f) = d, \quad \text{where } c, d \in \mathbb{N},\ c+d>0.$$

Based on the above analysis, we believe that $\max X^f$ and $\min X^f$ are closely related to a, b, c, and d. Subsequently, we proved the following result:

Lemma 2

For any $X \in \mathcal{P}_{fin,0}(\mathbb{Z})$ with $\min X = -x_-$ and $\max X = x_+$, we have

$$\min(X^f) = -c\,x_- - a\,x_+, \qquad \max(X^f) = d\,x_- + b\,x_+.$$

Properties of $\{0,1\}^f$

If Tringali and Wen's conjecture is correct, then for any $f \in \mathrm{Aut}\big(\mathcal{P}_{\mathrm{fin},0}(\mathbb{Z})\big)$, we naturally expect

$$\{0,1\}^f = \{0,1\} \quad \text{or} \quad \{0,-1\}^f = \{0,1\}.$$

Using the previous lemma, we indeed obtain this result.

Lemma 3

Let f be an automorphism of $\mathcal{P}_{fin,0}(\mathbb{Z})$. Then either

$$\{0,1\}^f = \{0,1\} \quad \textit{or} \quad \{0,-1\}^f = \{0,1\}.$$

The Case $\{0,1\}^f = \{0,1\}$

Now, without loss of generality, assume $\{0,1\}^f = \{0,1\}$.
If $\{0,-1\}^f = \{1,0\}$, define the map f_0 by $X^{f_0} = -X$. Then we have

$$\{0,1\}^{f_0 f} = \{1,0\}.$$

Thus it suffices to consider the case $\{0,1\}^f = \{0,1\}$ in our analysis.

Case: $\{0,1\}^f = \{0,1\}$

Assume $\{0,1\}^f = \{0,1\}$ for $f \in \mathrm{Aut}\big(\mathcal{P}_{\mathrm{fin},0}(\mathbb{Z})\big)$. According to the conjecture, we expect $X^f = X$ for all X.
Using the previous lemma, we first prove:

(1) $\{0,-1\}^f = \{0,-1\}$, and for any $X \in \mathcal{P}_{\mathrm{fin},0}(\mathbb{Z})$,

$$\min(X^f) = \min(X), \qquad \max(X^f) = \max(X).$$

Then, for any interval $[\![-a,b]\!] = a\{0,-1\} + b\{0,1\}$ with $a,b \in \mathbb{N}$:

$$[\![-a,b]\!]^f = a\{0,-1\}^f + b\{0,1\}^f = a\{0,-1\} + b\{0,1\} = [\![-a,b]\!].$$

Case: $\{0,1\}^f = \{0,1\}$ (Continued)

(2) For $X \in \mathcal{P}_{\text{fin},0}(\mathbb{N})$ (or $-X \in \mathcal{P}_{\text{fin},0}(\mathbb{N})$):

$$\min(X^f) = \min(X),\ \max(X^f) = \max(X) \text{ by (1)} \implies$$

$$f|_{\mathcal{P}_{\text{fin},0}(\mathbb{N})} \in \mathrm{Aut}\big(\mathcal{P}_{\text{fin},0}(\mathbb{N})\big).$$

Thus, by Theorem 1, we have:

$$X^f = X \quad \text{or} \quad X^f = \max X - X.$$

The case $X^f = \max X - X$ leads to a contradiction (by constructing a counterexample).

Remaining Case

We are now left with the final situation:

$$X \in \mathcal{P}_{\text{fin},0}(\mathbb{Z}), \quad \max(X) > 0 \text{ and } \min(X) < 0.$$

For this case, we use a tool called the **boxing dimension** of X ($\text{b.}\dim(X)$).
We prove $X^f = X$ by induction on $\text{b.}\dim(X) + \text{b.}\dim(X^f)$.
This method comes from Tringali and Yan's paper [2].

A Second Problem from Tringali and Yan

Our second problem also originates from a conjecture by Tringali and Yan: [Tringali and Yan] Let S be a monoid properly contained in $\mathbb{N}$. Then $\mathrm{Aut}\big(\mathcal{P}_{\mathrm{fin},0}(S)\big)$ must be the identity.

We do not address this conjecture directly, but instead consider an analogous problem:

Question 3

$$\mathrm{Aut}\big(\mathcal{P}_{\mathrm{fin}}(S)\big) = ?$$

where S is a **semigroup** properly contained in $\mathbb{N}$.

Main Results

We proved the following:

Theorem 4

Let $S \subsetneq \mathbb{N}$ be a semigroup.

(1) *If S is a* **discrete interval**, *i.e., $S = [\![k, \infty)$ for some $k \in \mathbb{N}$, then the only nontrivial automorphism of $\mathcal{P}_{fin}(S)$ is the involution*

$$\sigma(X) = \max X - X + \min X, \qquad \forall X \in \mathcal{P}_{fin}(S).$$

(2) *Otherwise, if S is* **not** *a discrete interval, then $\mathcal{P}_{fin}(S)$ has only the identity automorphism.*

Proof Strategy

Our proof proceeds in three steps:

1. **Step 1.** Let k be the least element of S such that $[\![k, \infty) \subseteq S$. We prove that $\mathcal{P}_{\text{fin}}([\![k, \infty))$ is stabilized by any automorphism f of $\mathcal{P}_{\text{fin}}(S)$. Thus, the problem reduces to studying the restriction $f|_{\mathcal{P}_{\text{fin}}([\![k,\infty))}$.
2. **Step 2.** Determine $\text{Aut}\big(\mathcal{P}_{\text{fin}}([\![k, \infty))\big)$.
3. **Step 3.** Based on the form of an arbitrary automorphism of $\mathcal{P}_{\text{fin}}([\![k, \infty))$, we complete the characterization of any automorphism f of $\mathcal{P}_{\text{fin}}(S)$.

We won't go into the proof details here. Finally, we have provided the preprint details in the references. Interested readers can download them from arXiv.

References

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Thanks!